The Legacy of Euclid

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In his *Elements,* Euclid brings to light many theorems through constructed propositions which form the birth of geometry as is taught around the world today. Euclid's work in his *Elements* not only form the basis of modern-day geometry, but is also observable in the works of later influential mathematicians. Though not all mathematicians reference his work, it becomes evident that much of what they discuss is derived from Euclid. Though Euclid may not be the initial discoverer of everything he has put in his *Elements,*—the works of the great Eudoxus are lost to the world— one must give him credit for his organization in this great work of his. From early studies on the motions of the universe to the beginning of calculus, even the work of Lobachevski can be traced back to Euclid's *Elements*.

Of all of the mathematical texts following Euclid which are read in the Integral Program, none is more heavily influenced by Euclid than Apollonius' *Conics*. Not only does Apollonius imitate Euclid in the structure of his propositions, by listing definitions, beginning a proposition with enunciating what he is setting out to prove, and then going step by step through his argument, but he even uses ideas from Euclid's *Elements* as logical argument to prove many of his steps. Almost every proposition in *Conics* involves at least one step involving a logical argument derived from Euclid. Were it not for Euclid's work, much of the work of Apollonius would not be possible. Though a great deal of his work hinges on the logical arguments that Euclid demonstrates in his *Elements*, to his credit, Apollonius does bring to light many things himself. Apollonius uses what Euclid has put forth to further develop the knowledge of geometry. The knowledge Apollonius brings forth about conic figures plays an essential role in later developments of calculus.

In order to exemplify how Apollonius both relies on Euclid, and also makes advancements in mathematics, the examples of the different types of diameters that Apollonius mentions is one of the major concepts introduced in Book I of Apollonius' *Conics*. Euclid gives a definition for the diameter of a circle in his very first book of his *Elements*. His seventeenth definition states:

> A *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle. (1)

The key similarity here between Euclid's definition, and the definitions that Apollonius gives is that a diameter bisects the figure it pertains to in some way.

There are three different definitions of a diameter used in Apollonius' *Conics*. At first we are merely familiar with the diameter of a circle from Euclid, but later, Apollonius' definitions and propositions give us different definitions of diameters for different figures. Three of the definitions in Book I are very important to the development of the term diameter; these definitions are definitions four through six. The propositions that correspond with Apollonius' definitions four through six are Proposition Eleven, Proposition Fourteen, and Proposition Sixteen, respectively. Each of these propositions reveals some new type of diameter.

The definition of a diameter that is revealed in Proposition Eleven defines the diameter of a parabola. The definition of a diameter of a parabola comes from Apollonius' definition of a curved line, which is his fourth definition in his first book. Apollonius' definition of a curved line states:

> Of any curved line which is in one plane, I call that straight line the diameter which, drawn from the curved line, bisects all straight lines drawn to this curved line parallel to some straight line; and I call the end of the diameter situated on the

> curved line the vertex of the curved line, and I say that each of these parallels is drawn ordinatewise to the diameter. (3)

Apollonius already gives the diameter of a curved line in the porism of Proposition Seven, so all that it is necessary for him to prove is that the curved line constructed in Proposition Eleven is a parabola. For this construction, please refer to Figure 1 in the Appendix. In Proposition Eleven, one is given any cone that is cut by a plane through its axis, thus making a triangle. The cone is also cut by another plane that cuts the base of the cone in a straight line which is perpendicular to the base of the axial triangle; this plane creates a curved section about the surface of the cone. The diameter of the section is to be constructed parallel to one of the sides of the axial triangle; this is a key part of the given statement because the diameter has to be parallel to one of the sides in order for the necessary conditions of a parabola to be discovered. There is to be a straight line drawn from the vertex of the section which is perpendicular to the diameter of the section; the length of this line must have the same relationship to the line between the vertex of the cone and the vertex of the section as the square on the base of the axial triangle has to the rectangle composed of the remaining sides of the axial triangle. By using the aforementioned proportion, the qualities of a parabola can be proven about the section.

With the information provided in the given, Apollonius proves that the square on an ordinate is equal in area to the rectangle contained by the corresponding abscissa and the line that was created perpendicular to the diameter of the figure from its vertex. Once this is proven, Apollonius says that the figure created with its diameter parallel to the side of the axial triangle is a parabola. Similarly, from the fact that we can take the diameter of a curved line, it can be proven that both a hyperbola and an ellipse can be found inside of a cone, as are proven in

Proposition Twelve and Proposition Thirteen of Book I. These propositions correspond to Definition Four, which is necessary in order to have the conclusions of each of these propositions. Definition Four defines the "diameter for any curved line". This definition includes a straight line that is drawn from a curved line and bisects all lines drawn parallel to some straight line. The point where this diameter meets the curved line is called the vertex, and the lines that were said to be parallel to some straight line are ordinatewise to the diameter. The diameters for parabolas, hyperbolas, and ellipses all have the conditions mentioned in Definition Four.

Another proposition that includes a definition of a diameter is Proposition Fourteen in Book I. For this construction, please refer to Figure 2 in the Appendix. This proposition gives us a transverse diameter. Apollonius' definition of a transverse diameter, which is included in his fifth definition, is stated as:

> Of any two curved lines lying in one plane, I call that straight line the transverse diameter which cuts the two curved lines and bisects all the straight lines drawn to either of the curved lines parallel to some straight line. (3)

In Proposition Fourteen, one is given a conic surface which is cut on both sides by a plane which does not go through the vertex. There are four steps necessary in order to prove what is concluded in this proof. The first is that the sections created by the plane that cuts the vertically opposite surfaces are both hyperbolas; this is necessary in order to prove the next step. The next step is that the diameter of both of the sections is one straight line. This must be proven because a transverse diameter has to be just one straight line. The third step is that the parameters of the two sections are equal to each other. This step is important in proving that the two sections are identical hyperbolas. The fourth step is that the transverse side of the figure is common.

Apollonius calls the two sections opposite. From the fact that the two sections are opposite, in the same plane, and also bisected by the same diameter, it can be said that the diameter which bisects them is a transverse diameter; this can be concluded because all of the requirements of a transverse diameter which are stated in Definition Five are held to be true about the diameter.

A third proposition that involves yet another definition of a diameter is Proposition Sixteen. For this construction, please refer to Figure 3 in the Appendix. This new diameter meets all of the requirements necessary in Definition Six for a line to be a "conjugate diameter". His sixth definition states:

> The two straight lines, each of which, being a diameter, bisects the straight lines parallel to the other, I call the conjugate diameters of a curved line and of two curved lines. (4)

In this proposition, one is given opposite sections (hyperbolas) with a transverse diameter. One is also given a line through the midpoint of the transverse side of the opposite sections, which is parallel to the ordinates of the two sections. Based on what is given, it can be proven that the aforementioned line through the midpoint of the transverse side is a diameter of the opposite sections, and that it is conjugate to the "transverse diameter".

There are three different types of diameters employed in Book I, but they all have similar properties. The common element shared among the three definitions of diameters that Apollonius gives is that they all bisect their corresponding figure in some way. By bisecting a figure, a diameter must pass through the figure, be in the same plane as it, and bisect the lines drawn ordinatewise (parallel to some straight line) to it. Whereas Euclid gives us a general idea of how a diameter is defined, Apollonius enhances the term diameter by showing that figures other than polygons can have diameters. Diameters are particularly important because they give us the necessary relationship between the square on an ordinate to the rectangle composed of the corresponding abscissa, and the parameter for parabolas, hyperbolas, and ellipses. Once the qualities of a diameter are known, many other things about parabolas, hyperbolas, ellipses, and cones can be proven. Apollonius later proves that there are infinite amount of diameters for the aforementioned figures.

Another mathematician who's work is influenced heavily by Euclid's *Elements* is Archimedes. Similar to Apollonius, Archimedes uses logical arguments derived from Euclid, and also goes about proving what he sets out to in the same way Euclid does.

One of Archimedes' propositions that stands out as having a significant amount derived from Euclid is his proposition that includes the method of exhaustion. Archimedes' method of exhaustion incorporates Euclid's first proposition from his tenth book of his *Elements.* Euclid's enunciation states:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. (237)

In this proposition, Euclid is able to prove that it is possible to divide a magnitude infinitely without limit. The way in which this concept is used by Archimedes is to demonstrate how it is possible to determine the area of a circle based on the relationship it has to a certain triangle. Archimedes' proof is a *reductio*, a method of proving something indirectly which Euclid often does. In other words, he demonstrates that because the circle is neither greater nor less than the area of the given triangle, it must be equal in area to it.

Archimedes demonstrates that it is possible to prove that the area of any circle is equal to that of a right-angled triangle which has one of its sides equal to the radius of the circle and another equal to the circle's circumference. In his enunciation, he states:

> The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle. (91)

Refer to Figure 4 in the Appendix for the construction of this proof. The proof is a reductio, so Archimedes eliminates both of the other two possibilities. He first eliminates the possibility that the area of the circle is greater than that of the given triangle. In order to do this, Archimedes supposes that the circle is greater in area than the aforementioned triangle. He then inscribes a square in the circle, and divides the arcs created between the points of contact of the square and the circle in half. Once he has created these points which divide the arcs in half, he connects them to the points of contact between the square and the inscribed circle, thus creating an inscribed rectilinear figure. The difference between the newly inscribed rectilineal figure and the circle consists of eight segments. Archimedes says to let the sum of all of these segments be less than the difference in area of the circle and the triangle. The rectilineal figure is therefore greater in area than the given triangle. This is due to the fact that the amount that the circle exceeds the triangle is greater than the amount that it exceeds the rectilineal figure, or the sum of the eight segments leftover.

To explain this in another way, if a circle is greater than a second area by a certain amount, and is greater than a third area by an amount which is less than that by which it is greater than the second area, the second area will be less than the third area. This is because there is a greater difference between the circle and the second area than there is between the circle and the third area, making the third area closer to being equal to the circle than the second area.

The next thing Archimedes does is take the center of the circle and drop a perpendicular from it to one of the bases of the segments he created. He then says that the line created is less than the side of the triangle, meaning the side equal to the radius of the circle. It is less than the radius of the circle because it falls short of reaching the circumference of the circle. Therefore, because it is less than the radius of the circle, it is also less than the side of the triangle which was created equal to the radius of the circle. Archimedes also shows that the perimeter of the rectilineal figure is less than the base of the given triangle, since the base of the given triangle is equal to the circumference of the circle, and the rectilineal figure is inscribed in the circle. Because of the fact that the perimeter of the rectilineal figure is less than the base of the given triangle, and the height of the triangles that make up the rectilineal figure was proven to be less than the other side of the triangle, the area of the rectilineal figure is, therefore, less than that of the triangle. This is absurd due to the fact that it was said to be greater than the triangle earlier in the proof. Archimedes similarly proves that the circle cannot be smaller in area than the given triangle by circumscribing a square about the circle. By proving that the area of the circle can be neither greater nor smaller in area than the given triangle, Archimedes has indirectly shown that the circle must be equal in area to it by means of a *reductio.*

The main idea one is supposed to take from this proof is that as long as the sum of the difference between the rectilineal figure and the circle is less than the difference between the triangle and the circle, the segments can be divided an infinite amount of times. This is what Archimedes seems to mean by “method of exhaustion.” In order for this method to hold true, Euclid's proposition claiming that it is possible to divide a magnitude infinitely without limit must also be true.

Though it is essential for Euclid's first proposition in his tenth book to be true in order for Archimedes' proposition on the method of exhaustion to be valid, there is another proposition in Euclid's *Elements* that makes evident that Archimedes derives the main arguments in his proposition on the method of exhaustion from Euclid. That proposition is Euclid's second in his twelfth book. Archimedes sets up his proposition in the same manner as Euclid's proposition.

In the enunciation of his second proposition in his twelfth book, Euclid states:

> Circles are to one another as the squares on the diameters. (412)

For the construction of this proof, please refer to Figure 5 in the Appendix. Euclid sets up this proposition such that one is given any two circles of different size. He sets out to prove that as the first circle is to the second, so is the square on the diameter of the first to the square on the diameter of the second. Euclid sets up his proof in the form of a reductio. He states that if the ratio between the two circles and the squares on their diameters does not hold true, there are only two other possible cases, namely, that the ratio of the square on the diameter of the first circle to the square on the diameter of the second will equal the ratio that the first circle has to an area which is either greater or less than that of the second circle. He supposes first that the first circle is in ratio to an area less than that of the second circle. Euclid inscribes a square in the second circle, and bisects the circumference of the segments of the circle contained by the circumference and the sides of the square. Then, inside of the four segments, he connects the points bisecting the segments to the points where the square intersects the circle, creating triangles. Euclid says that if one were to circumscribe a square about the second circle, with the sides being tangents and their points of intersection being the same points that the inscribed square intersects the circumference, that it is possible to prove that the inscribed square is greater in area than half the circle. This is due to the fact that the inscribed square is equal to half of the circumscribed square, and since the circumscribe square is greater than the circle, the inscribed circle would be greater than half of the circle. He also proves, by the same argument, that if one were to circumscribe rectangles about the segments of the circle by drawing tangents from the points of on the circumference of the circle which bisect the segments, that the triangles contained by the segments will each be greater than half of the segment they correspond with. Once this has been shown, it is obvious from Euclid's first proposition in his tenth book that by repeating this process of bisecting the circumferences of the segments that remain continually, that there will eventually be an area made up of the sum of the segments between the circle and the inscribed polygon that is less in area than the amount by which the area of the second circle exceeds the area which the first circle was said to be in ratio to. Euclid ultimately goes on to prove what he set out to, but it is this particular section of his proof in which Euclid inscribes a square in a circle, and continually bisects the arcs of the segments created which Archimedes uses in his proof.

In his "Geometric Proof of the Fundamental Theorem," Isaac Barrow demonstrates how differentiation is related to integration. In other words, he shows how areas under curves are related to the slopes of the tangents they correspond with. He is combining the two main ideas of calculus. Prior to this in the manual, these two concepts have been discussed separately. By doing this, Barrow has opened things up for further investigation. The Fundamental Theorem is one of the most important discoveries involved in calculus, however, the discovery of this theorem would not have been possible without the works of Euclid. Throughout this proof, there are many steps which are derived from Euclid's propositions.

In his demonstration, Barrow constructs a geometric figure in order to illustrate how his theorem works. First, he constructs any curve, gives it an axis, and then constructs ordinates perpendicular to the axis which increase continually as the curve increases. The next step in his construction involves taking another curvilinear line that has a certain relationship to the first curvilinear line. The relationship of the two curves is as follows: if one is to take any straight line perpendicular to the axis, which cuts both the axis and the two curves, the rectangle contained by the segment of the perpendicular line which lies between the second curve and the axis and another given length, is equal to the area enclosed by the axis, the original curve, and the line perpendicular to the axis. Barrow then states that the ratio of the line segment cut off between the first curve and the axis to the line segment cut off between the second curve and the axis is equal to the ratio that the given length has to a length on the axis cut off between the perpendicular line and some other point. This other point lies between the first ordinate, and the line perpendicular to the axis. The editor makes note that this point and the point where the perpendicular meets the second curve should be connected, as Barrow left this out but goes on to mention this line as a part of what must be proven. Both the line of given length and the segment on the axis are determined by the ratio they have to each other, which must be equal to the ratio which the ordinate has to the rest of the perpendicular line.

Now that he has laid out the construction of his diagram, Barrow begins his proof. He claims that the aforementioned line segment, which lies between the point where the perpendicular touches the second curve and the point on the axis created by the proportion it holds to the given length, will touch the second curve. In order to prove this, a point is taken anywhere on the second curve. The first demonstration involves taking it on the side of where the perpendicular meets this curve which is closest to the beginning of the axis. Through this point, a line is drawn parallel to the first ordinate. Another line is drawn through this point parallel to the axis, cutting both the line said to touch the second curve in the statement which is to be proven, and the first mentioned perpendicular line. With these two lines constructed, Barrow is able to state a string of proportions. The first part of the proportion comes from Euclid's Book VI, Proposition Four. For this proposition, refer to Figure 6 in the Appendix. In this proposition, the enunciation reads:

> In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles. (126-127)

In Barrow's diagram, equiangular triangles are created because there is a triangle created inside of another larger triangle by a line which cuts the larger triangle, and is perpendicular to one of its sides. Therefore, the corresponding sides of these two triangles are proportional. The second proof involved in the string of proportions comes from the original proportion that was derived from the construction of the figure. What Barrow is saying here is that the second term is to the fourth term as the first term is to the third term, which comes from Euclid's Book VII, Proposition 13. In this proposition, Euclid states:

> If four numbers be proportional, they will also be proportional alternately. (166-167)

This is simply a more general way of saying that if four numbers are proportional, the first term will be to the third term as the second term is to the fourth, which is exactly what Barrow is saying in his proof. A proposition from Euclid is necessary for the third part of the proportion as well. In this part of the proportion, Barrow substitutes the first ratio into the last proportion, and then simply multiplies the means by the extremes.

The proposition from Euclid that this step comes from is Euclid's sixteenth proposition of his sixth book, of which the enunciation states:

> If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional. (136-137)

Barrow's proof involves the first part of this enunciation, since he has already proven that four straight lines are proportional. Because of Euclid's proof, the area created by the means can be said to be proportional to the area created by the extremes for any four straight lines which are proportional, thus allowing Barrow to use this in his proof. Euclid's propositions play an important role in Barrow's discovery of the Geometric Proof of the Fundamental Theorem.

The next step in the proof involves multiplying the length which is derived from the construction of the diagram by the line segment of the original line that was drawn perpendicular to the axis, which lies between the second curve and the line drawn parallel to the axis. The area of multiplying these two lines is said to be equal to the area contained by the original curve, the axis of the original curve, the line that was drawn perpendicular to this axis, and the newly created perpendicular line, which starts at the point taken on the second curve. This step involves subtracting things which are proportional from what has been stated to be equal, namely the rectangle contained by the segment of the perpendicular line which lies between the second curve and the axis and another given length, being equal to the area enclosed by the axis, the original curve, and the line perpendicular to the axis. By a substitution of terms, and showing that one area is less than another, the most important step of the proof is revealed. This comes from the fact that there is a constant side for two of the areas in the inequality, which means that the other two sides can be said to be unequal as well. From this inequality, it is evident that the line between the point where the perpendicular to the axis meets the second curve, and the point on the axis which is found by the proportion original proportion, can be said to lie under the second curve. The way in which integration and differentiation can be said to be related from the diagram is by the fact that the slope for the function of a curve at a point taken on it holds a certain relationship to the area contained by another curve (which represents the derivatives of the first curve), the axis of the first curve, and the ordinate of this other curve.

This note has essentially shown how integration and differentiation, the two major ideas that make up calculus, are related to each other. The works of past mathematicians are vital for the proof in this note to be considered valid, as is evident from the fact that it uses so many of Euclid's propositions. All of the previous ideas of the manual are brought together in this note, just as the works of many preceding mathematicians have built off of each other.

Newton's first lemma is a vital part of his first book of his *Principia, The Central Argument*. In his first Lemma, Newton shows one what it means for both two quantities and also two ratios to become ultimately equal. In his enunciation for his first lemma, Newton states:

> Quantities, as well as the ratios of quantities, which in any finite time you please constantly tend towards equality, and before the end of that time approach nearer to each other than by any given difference you please, become ultimately equal. (47)

Newton gives one two quantities constantly tending towards equality in any given finite time. He proves that as the time limit is reached the difference between the two quantities will be infinitely small, and that because this difference is smaller than any give difference, the two quantities cannot be said to be unequal. If the two quantities cannot be said to be unequal, they are therefore ultimately equal. The concept of being 'ultimately equal' is important because it is used throughout the rest of Book I to prove various things. The concept of ultimately equality is a necessary part in Newton's overall discovery of the law of gravity. The rest of the lemmas in Book I use ultimate equality to prove different things about curved lines. These lemmas lead up to Proposition 1.

Newton's first lemma in Book I is very similar to Euclid's first proposition in Book X of his *Elements,* the same proposition from which Archimedes derives his proof where he includes the method of exhaustion, as was discussed earlier. His argument is actually the inverse of Euclid's. Euclid's enunciation states:

> Two unequal magnitudes being set out, if form the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. (237)

The enunciation in Euclid's proposition gives one two unequal magnitudes. What Euclid proves about these magnitudes is that if from the greater given magnitude, there is subtracted a magnitude greater than its half, and from the magnitude leftover from this division, a magnitude greater than its half is subtracted, and if this process is repeated continually, there will eventually be some magnitude less than the smaller given magnitude. In other words, Euclid proves through this process that one is left with a magnitude which is less than the lesser given magnitude.

The enunciation in Newton's lemma gives one two quantities that are constantly tending towards equality in any given finite time. What Newton proves in the lemma is that just as the finite amount of time is expiring, the two quantities become "ultimately equal." Newton's lemma seems to use Euclid's proposition in that it has very similar concepts involved in its proof. Euclid's proof, however, is much more comprehensible than Newton's in that his proof is written in a much clearer fashion. Where Euclid uses a diagram with steps that are separated into the order of their logical progression, Newton merely shows that because of the fact that the quantities cannot be said to be unequal, they must therefore be equal.

Euclid would approve of Newton's proof because of the fact that his second proposition in his Book XII of his Elements uses the exact same logical argument. To restate the enunciation of Euclid's XII.2, as was used earlier to show similarities to Archimedes' proposition on the measurement of a circle, Euclid states:

> Circles are to one another as the squares on the diameters. (412)

Both Euclid's second proposition from his twelfth book, and Newton's first lemma are indirect proofs. Since the difference between the two quantities that Newton is left with is infinitely small, there is no quantity that is smaller than it that would make the quantities unequal, and they must therefore be said to be equal. This incorporates the method of exhaustion which has been used by both Euclid and Archimedes.

Although Newton's first lemma of his first book is a small step towards what he ultimately ends up proving, it is still extremely important. This proposition is vital for Newton's proof of universal gravitation. In this proposition, Newton proves that bodies that are drawn to an unmoving center of forces and are also in an unmoving plane, make areas that are proportional to the times traversed. In order to prove this, Newton describes the path that the body would traverse in the same time interval as any given first time interval, but without a force acting upon it; the two triangles created are equal because they are under the same height and of equal bases. Newton then goes on to prove that if the body were acted upon by a centripetal force during the second time interval, that it would create a triangle equal to the triangle in which there was no force acting on the body because of equal bases and equal heights. He then states that this can be proven for any number of triangles. If the number of triangles increased infinitely, and their breadth decreased in infinitum, the ultimate perimeter created would be a curve; from this one can say that if the centripetal force is constant, that the body's path will be curvilinear. One can only say that the perimeter would be curvilinear and also a circle if the term 'ultimately equal' is a legitimate concept. Because of the fact that the triangles in the proof could not be said to become curvilinear without the term 'ultimately equal' having validity, one would not be able to prove that a body is capable of moving in a circle. Without circular motion, there is no such thing as centripetal force, and centripetal force is essential in Newton's proof of the law of gravity.

What Newton is trying to prove in his sixth proposition is that for a body revolving in any orbit, about an immobile center, which creates any arc that is nascent, that the force will vary ultimately as the sagittae divided by the time squared. In other words, he is trying to prove that the force varies directly as the sagittae in the ultimate case, and inversely in the duplicate ratio of the time. The duplicate ratio comes from Euclid's ninth definition in his fifth book, which states:

> When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second. (99)

His first step involves the fourth corollary to the first proposition, which demonstrates that the ratio of the sagittae varies ultimately as the ratio of the forces for a body at two different points on an orbit. His next step involves the third corollary to the eleventh lemma, which states that the sagittae is in duplicate ratio of the time.

Newton's next step involves Euclid's Book VI, Proposition 23, which states that:

> Equiangular parallelograms have to one another the ratio compounded of the ratio of their sides. (145)

Refer to Figure 7 in the Appendix for the construction of this proposition. This is related to Newton's sixth proposition in that Newton uses the sagittae, the forces, and the time intervals to represent the sides of parallelograms, and compounds the ratios of the first two steps. By then using substitution, and also compounding both sides of this new ratio with the inverse ratio of the times squared, Newton finally proves what he set out to. The concept of 'ultimate equality' is used in this proposition in order to prove how the relationship between the force and the time varies. This relationship is important for discovering the law of centripetal force that is found in the next proposition.

In Proposition 17, the last proposition in the first book, all of the information from the previous propositions is used to find a conic section's foci, latus rectum, and major axis. Then, Newton goes through each of the conic sections--ellipse, hyperbola, and parabola--and demonstrates this. This is a necessary proof in order to begin the actual calculations of planets and their motions.

The first few propositions in Book I prove different qualities of bodies moving in curvilinear orbits. Propositions nine through thirteen give us the different force laws for the various types of curves. Propositions fourteen through sixteen use these force laws to build up to Proposition 17, each also using the former proposition in its proof. The propositions leading up to seventeen seem to start out with an equation, and build upon that equation with every proposition. Proposition 17 uses all that has been learned in the previous propositions and applies it to an actual body moving in an orbit for the cases of different curvilinear paths. For the construction of Proposition 17, refer to Figure 8 in the Appendix. None of this would be possible, however, without the concept of 'ultimately equal.' This concept is the foundation for all of the other things that Newton proves because it is used in many of the propositions in Newton's first book. From the propositions involving the concept of 'ultimate equality,' the propositions build upon each other to work towards Newton's law of gravity.

All that has been addressed thus far has been in support of my earlier claim that Euclid's *Elements* is a foundation for western mathematics. There are some mathematicians, though, who's work may seem to contradict the work of Euclid. A prime example of this can be seen in Lobachevski's "Theory of Parallels." Lobachevski's sixteenth theorem in particular could be seen as a counterargument against Euclid's fifth postulate. A construction for this theorem has been included in the Appendix (see Figure 9). Euclid's fifth postulate states:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (2)

Though Euclid already gives a definition of parallel lines prior to stating Postulate Five, the importance of his fifth postulate is to state that no two straight lines, being intersected by a third and having the sum of their consecutive interior angles less than two right angles will be parallel.

What Euclid is hypothesizing is that in the relationship between three straight lines which he gives in his fifth postulate, the two straight lines intersected by a third can *only* be parallel if the sum of the consecutive interior angles be equal to two right angles. Based on the fact that it is impossible for a finite segment of a straight line to be constructed, so much as an infinite straight line, it is evident that the statements of both Euclid and Lobachevski on parallel lines are merely assumptions.

Lobachevski's sixteenth theorem calls Euclid's fifth postulate into question because of the fact that it is impossible for a human being to perceive two straight lines produced indefinitely. He is not necessarily refuting Euclid's claim, but rather noting the uncertainty of it. This is evident from the sentence in Lobachevski's demonstration where he states, "In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for example AG, which do not cut DC, how far soever they may be prolonged" (13). In his theorem, Lobachevski shows a way in which it may be possible for two straight lines, being intersected by a third, and having the sum of their consecutive interior angles less than two right angles, to be considered parallel to each other. Were this theorem to be held as true, it would contradict the hypothesis Euclid makes in his fifth postulate. Due to the fact that both Euclid's fifth postulate, and Lobachevski's sixteenth theorem are based on assumptions, since it is impossible to produce lines indefinitely, one cannot say definitively that either of their hypotheses are correct. However, when using imperfect representations of geometric figures to conceptualize Euclid's propositions, everything that is included in his *Elements* is valid as far as we know.

Over the course of reading this essay, I hope you, the reader, did not get too perplexed in trying to understand the various propositions and theorems I have included. The purpose of the various propositions and theorems was to elucidate how other mathematicians have used the axiomatic method implemented by Euclid, as well as to show specific logical arguments these mathematicians have derived from him. Each of the propositions of Archimedes, Barrow, and Newton involve indirect proofs, a style of proof implemented by Euclid. Indirect proofs are important because they allow one to prove something which would be simply impossible to demonstrate by means of a direct proof. In an indirect proof, something is proven by showing that all of the cases which would contradict what is set out to be shown are false. With that said, I hope it has been made clear that though he may have lived thousands of years ago, Euclid's work lives on in the mathematics taught in classrooms around the world today through not only his work on geometry, but also that of the mathematicians to follow him.

Appendix

Fig 1

Fig 2

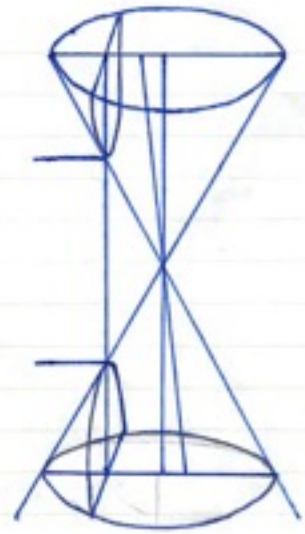


Fig 3

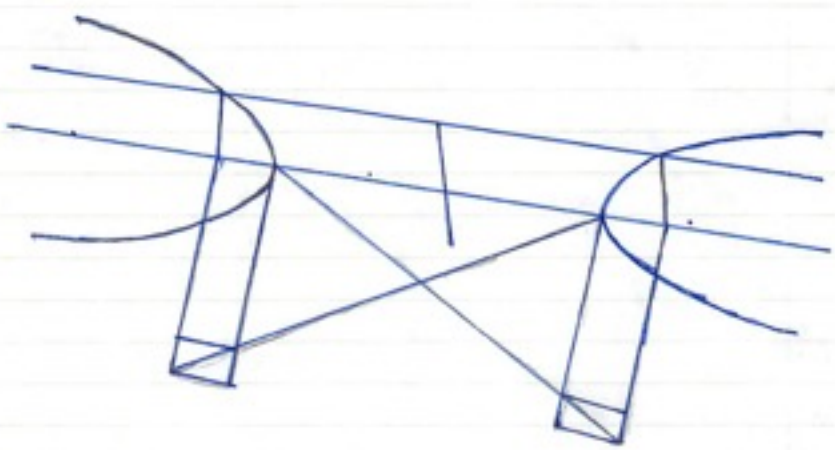
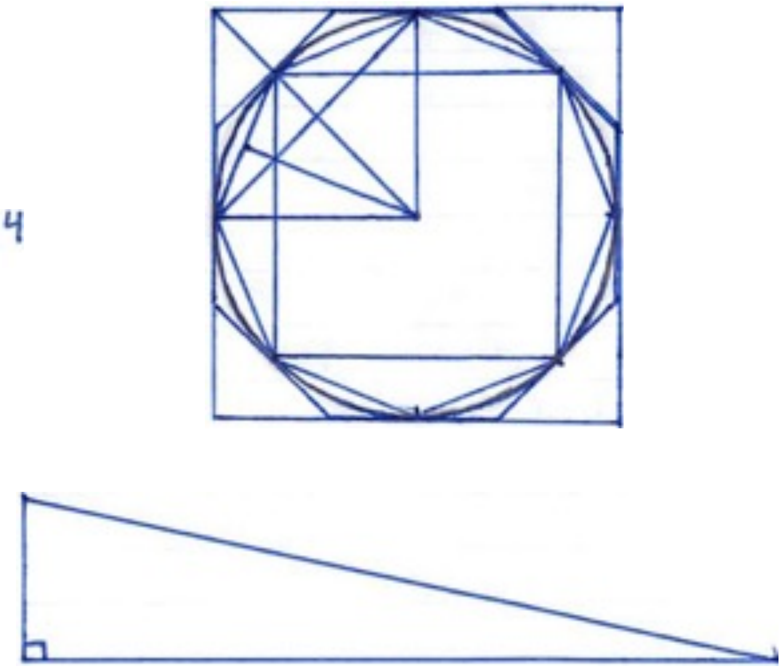
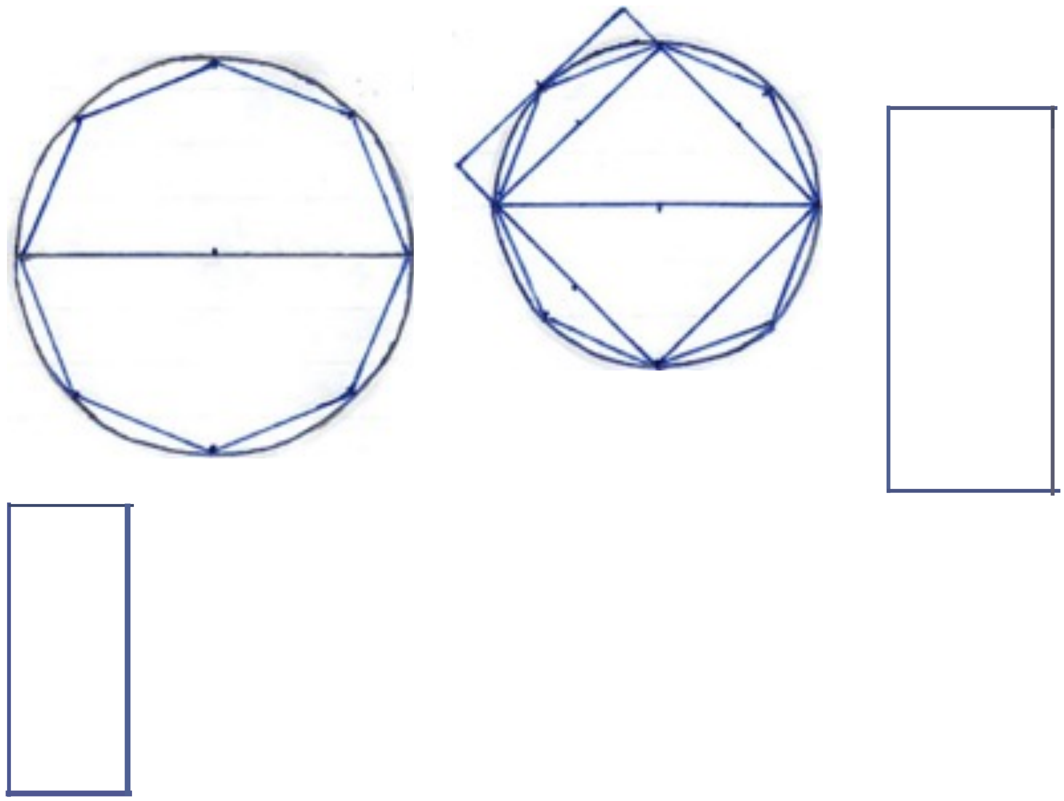


Fig. 4





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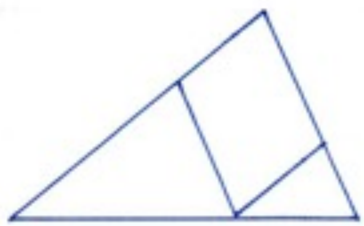

Fig. 6

Fig. 7

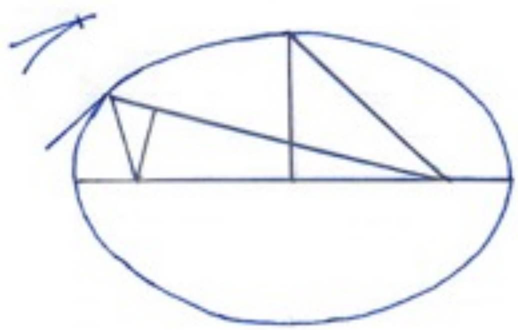

Fig 8

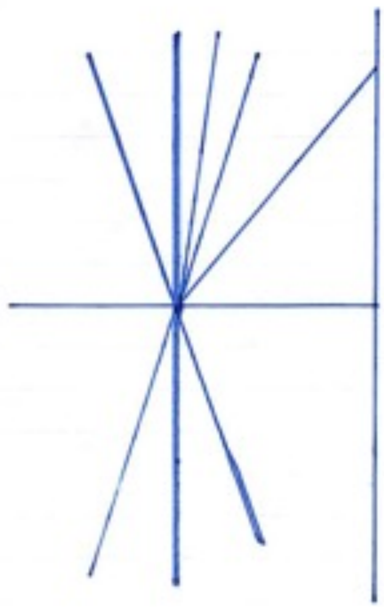

Fig. 9

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